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On the Khinchin Constant

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Abstract

We prove known identities for the Khinchin constant and develop new identities for the more general Hölder mean limits of continued fractions. Any of these constants can be developed as a rapidly converging series involving values of the Riemann zeta function and rational coefficients. Such identities allow for efficient numerical evaluation of the relevant constants. We present free-parameter, optimizable versions of the identities, and report numerical results.

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1 Introduction

The Khinchin constant arises in the measure theory of continued fractions. Every positive irrational number can be written uniquely as a simple continued fraction $[a_0, a_1, a_2, \cdots, a_n, \cdots]$; i.e., with a_0 a non-negative, and all other a_i positive integers. The *Gauss-Kuz'min distribution* ([11]) predicts that the density of occurrence of some chosen positive integer k in a random such fraction is given by

$$\mathrm{Prob}(a_n = k) = -\log_2\left[1 - \frac{1}{(k+1)^2}\right].$$

In his celebrated text, Khinchin ([11]) uses the Gauss-Kuz'min distribution to show that for almost all positive irrationals the limiting geometric mean of the positive elements a_i of the relevant continued fraction exists and equals

$$K_0 := \prod_{k=1}^{\infty}\left[1 + \frac{1}{k(k+2)}\right]^{\log_2 k} = \prod_{k=1}^{\infty} k^{\log_2\left[1+\frac{1}{k(k+2)}\right]}.$$

The fundamental constant K_0 is the *Khinchin constant.* It is known that this constant can be cast in terms of various converging series, the following example of which having been used decades ago to provide the first high-precision numerical values for K_0 [16, 21, 22]

$$\log(K_0)\log(2) = \sum_{s=1}^{\infty}\frac{\zeta(2s)-1}{s}\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{1}{2s-1}\right) \quad (1)$$

This series can be rendered even more computationally efficient via the introduction of a free integer parameter. We used a carefully optimized free-parameter series to resolve K_0 to over 7000 decimal places ($K_0 = 2.68545200106\cdots$ see Section 5).

The Khinchin constant can be thought of as a special case of a more general statistical mean. For any real number $p < 1$, the *Hölder mean* of order p of the continued fraction elements, namely $\lim_k[(a_1^p + a_2^p + \cdots + a_k^p)/k]^{1/p}$, also exists with probability one and equals

$$K_p := \left\{\sum_{k=1}^{\infty} -k^p \log_2\left[1 - \frac{1}{(k+1)^2}\right]\right\}^{1/p}.$$

(See the final section of Khinchin's book ([11]) for a proof for $p < \frac{1}{2}$, or more modern references on ergodic theory for a proof for $p < 1$ [15].) We may interpret K_0 as the limiting instance of the K_p definition as $p \to 0$. We shall show that for any negative integer p the Khinchin mean of order p satisfies an identity

$$(K_p)^p \log(2) = \sum_{s=2}^{\infty}(\zeta(s-p) - 1)Q_{sp} \quad (2)$$

where each coefficient Q_{sp} is rational. Again there is a free-parameter generalization, which we employed to resolve the *harmonic mean* K_{-1} also to over 7000 decimal places ($K_1 = 1.74540566240 \cdots$ see Section 5). It is of interest that, evidently, only K_0 can be written as a series involving exclusively even zeta arguments. The computational implications of this unique property of K_0 are discussed in Section 5. We should mention that aside from the Shanks-Wrench series for K_0 there are other previously known formulae for Khinchin means, some of which formulae involving derivatives of the zeta function [19].

2 Fundamental Identities

This section is devoted to presenting the basic identities. We begin with a list of preliminary, largely elementary, results needed in the paper. All of these rearrangements may be easily justified.

Lemma 1. (a)

$$-\log(1-x)\log(1+x) = \sum_{k=1}^{\infty} \frac{A_k}{k} x^{2k}$$

where $A_s := \sum_{k=1}^{2s-1} (-1)^{k-1}/k$.

(b)

$$\sum_{k=2}^{N} \log(1-\frac{1}{k})\log(1+\frac{1}{k}) - \sum_{k=2}^{N} \log(k-1)\log(1-\frac{1}{k^2}) = -\log(N)\log(1+\frac{1}{N}).$$

Thus, (c)

$$\sum_{k=2}^{\infty} \log(1-\frac{1}{k})\log(1+\frac{1}{k}) = -\log(K_0)\log(2).$$

Proof. Part (a) is most easily seen by differentiating both sides. The left-hand side becomes $f(x) - f(-x)$ where $f(x) := \log(1+x)/(1-x)$. Using the standard relationship

$$\sum_{k=1}^{\infty} \frac{a_k}{1-x} x^k = \sum_{k=1}^{\infty} \{\sum_{j=1}^{k} a_j\} x^k$$

produces (a).
Part(b) is easily established inductively after expanding the left-hand side.
Part (c) follows on taking limits and noting that

$$-\sum_{k=2}^{N} \log(k-1)\log(1-\frac{1}{k^2}) = \log(K_0)\log(2),$$

as follows from the definition of K_0. ☺

We will find it convenient to use the *Hurwitz zeta function* written

$$\zeta(s,N) := \sum_{n=1}^{\infty} \frac{1}{(n+N)^s}$$

so that $\zeta(s) = \zeta(s,0)$ and so that for N a non-negative integer

$$\zeta(s,N) = \zeta(s) - \sum_{n=1}^{N} \frac{1}{n^s}.$$

With this notation we have:

Lemma 2. (a) For N a positive integer

$$\sum_{n=1}^{\infty} \zeta(n,N) = \frac{1}{N}.$$

(b) For N a positive integer

$$\sum_{n=1}^{\infty} \frac{\zeta(2n,N)}{n} = \log(N+1).$$

(c)

$$\int_0^1 \frac{\log(1-t^2)}{t(1+t)} dt = -\log^2(2).$$

Proof. The proofs of the first two identities are similar and rely on expanding the zeta terms, rearranging the order of summation and re-evaluating. In both cases, the result telescopes to the desired conclusion.

Part (c) is less immediate. Actually, the indefinite integral is evaluable with the aid of the dilogarithm ([12]). Alternatively, Maple yields the result that

$$\int_0^t \frac{\log(1-x^2)}{x(1+x)} dx$$

equals the log terms

$$-\frac{\log^2(1+t)}{2} - \log^2(2) + \log(2)\log(1-t) - \log(1+t)\log(1-t) + \log(t)\log(1-t)$$

plus the dilog terms

$$\operatorname{dilog}(t) - \operatorname{dilog}(1+t) - \operatorname{dilog}(\frac{1+t}{2}),$$

and Maple also happily performs the final evaluation (the limit at 1). ☺

We are now in a position to establish the general Shanks-Wrench identity [16] for K_0.

Theorem 3. For any positive integer N,

$$\log(K_0)\log(2) = \sum_{s=1}^{\infty} \zeta(2s, N)\frac{A_s}{s} - \sum_{k=2}^{N} \log(1-\frac{1}{k})\log(1+\frac{1}{k}). \tag{3}$$

where $A_s := \sum_{k=1}^{2s-1}(-1)^{k-1}/k$.

Remark. N is a free parameter that can be optimized in actual computations to significantly reduce the number of zeta evaluations required. Variation of this parameter also provides a kind of error check, for whatever the choice of positive integer N, one expects an invariant result for the left-hand side. Note that in the case $N = 1$ the second summation is empty, and we recover precisely the K_0 identity (1) of Section 1.

Proof. Let $f(N)$ denote the right-hand side of (1). Then

$$f(N-1) - f(N) = \sum_{s=1}^{\infty} \frac{A_s}{s} N^{-2s} + \log(1-\frac{1}{N})\log(1+\frac{1}{N})$$

which equals zero by Lemma 1(a). Thus, since $\zeta(2s, N) \to 0$, sufficiently rapidly

$$\begin{aligned} f(1) = f(N) &= \lim_{N\to\infty} f(N) \\ &= -\sum_{k=2}^{\infty} \log(1-\frac{1}{k})\log(1+\frac{1}{k}). \end{aligned}$$

By Lemma 1(c), this sum agrees with $\log(K_0)\log(2)$. ☺

As a companion relation to the identity of Theorem 3, we can establish an elegant integral representation for the left-hand side. There is a powerful generalization of Lemma 2(b) in the form of a generating function based on Euler's product for $\sin(\pi t)/(\pi t)$ (see [18], page 249). For real t in [0,1) define $g(t)$ by

$$g(t) := \sum_{s=1}^{\infty} \frac{\zeta(2s)-1}{s} t^{2s} = -\log\left(\frac{\sin(\pi t)}{\pi t}\right) + \log(1-t^2), \tag{4}$$

and define also the limiting case $g(1) := \log 2$. We only need observe now that, on the basis of Theorem 3, with parameter $N = 1$,

$$\log(K_0)\log(2) = \int_0^1 \frac{\log(2) + g(t)/t}{1+t} dt,$$

and with the help of the previous dilogarithm integral evaluation we thus arrive at an integral representation. ([16] contains an equivalent, though less streamlined, integral identity.)

Corollary 4. The following integral representation holds for K_0:

$$\int_0^1 \frac{\log[\sin(\pi t)/(\pi t)]}{t(1+t)}dt = -\log(K_0)\log(2).$$

The integral is sufficiently easy to handle that Greg Fee has computed 500 digits of K_0 in Maple this way in about 10 minutes. It is amusing to observe that Lemma 1(c) may also be turned into an an analogous integral form:

$$\log(K_0)\log(2) = \int_1^\infty \frac{\log(\lfloor t \rfloor)}{t(1+t)}dt = \int_0^1 \frac{\log(\lfloor 1/t \rfloor)}{1+t}dt.$$

This was observed from a very different starting point by Robert Corless [9] but follows immediately on breaking the first integral up at integer points.

We now derive new, corresponding identities for the higher-order Khinchin means. They are in some sense simpler, since one logarithmic term is replaced by a negative integral power. There is an observation that leads directly to a zeta function expansion for these general Khinchin means. Note that a sum of terms $k^p \log(1-(k+1)^{-2})$ can be expressed, via expansion of the logarithm, in terms of sums of the form (note p is assumed to be a negative integer):

$$\sum_{n=2}^\infty \frac{1}{n^{2s-p}(1-1/n)^{-p}}.$$

Upon expansion of the term

$$1/(1-1/n)^{-p}$$

in powers of $1/n$, we obtain an identity for the p-th power of K_p as a series of zeta functions. The result, after the same free parameter manipulations we used for K_0, reads:

Theorem 6. For negative integer p and positive integer N we have

$$K_p^p \log(2) = \sum_{n=1}^\infty \frac{\sum_{j=0}^\infty \binom{j-p-1}{-p-1}\zeta(2n+j-p, N)}{n} - \sum_{k=2}^N \log(1-\frac{1}{k^2})(k-1)^p.$$

Remark. Note that for $N = 1$ the final sum is empty, the coefficient of any given $\zeta(s)$ is an easily computed rational, and we immediately establish a general series with rational coefficients, (2) of Section 1.

Corollary 7. The harmonic Khinchin constant satisfies for integer $N > 0$:

$$\frac{\log(2)}{K_{-1}} = \sum_{n=1}^\infty \frac{\frac{1}{N} - \sum_{k=2}^{2n}\zeta(k, N)}{n} - \sum_{k=2}^N \frac{\log(1-k^{-2})}{k-1}.$$

Proof. It suffices to show that

$$\sum_{j=0}^{\infty} \zeta(2n+j+1, N) + \sum_{k=2}^{2n} \zeta(k, N) = \frac{1}{N}.$$

This follows from Lemma 2(a). ☺

3 POLYLOGARITHMS AND RELATED ZETA FUNCTION IDENTITIES

There exist some interesting identities for the Khinchin constant in terms of polylogarithm evaluations. One particularly interesting polylogarithm identity is obtained by resolving the integral representation of Corollary 4 in polylogarithm terms [23]. One may employ the Euler product for $\sin z/z$ to write the integral as a sum of logarithmic integrals, each in turn expressible in terms of polylogarithms. This procedure leads to the series:

$$\log(K_0)\log(2) = \log^2(2) + \mathrm{Li}_2(-\frac{1}{2}) + \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n \mathrm{Li}_2(\frac{4}{n^2})$$

where, the *polylogarithm* $\mathrm{Li}_m(z) := \sum_{k=1}^{\infty} z^k k^{-m}$.

A more direct application of polylogarithms is to invoke the classic Abel identity:

$$\log(1-x)\log(1-y) = \mathrm{Li}_2(\frac{x}{1-y}) + \mathrm{Li}_2(\frac{y}{1-x}) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \mathrm{Li}_2(\frac{xy}{(1-x)(1-y)})$$

to Lemma 1(c), with $x := 1/n, y := -1/n$ to obtain, from a telescoping sum:

$$\log(K_0)\log(2) = \frac{\pi^2}{6} - \frac{1}{2}\log^2(2) + \sum_{n=2}^{\infty} \mathrm{Li}_2(\frac{-1}{n^2-1}).$$

An interesting line of analysis starting from this last polylogarithm series is to "peel off" parts of the Li_2 function, casting the corrections in closed form. Such a procedure gives polylogarithm-based analogues of Theorem 3. For example, one can replace the last Li_2 summand above with a more rapidly decaying term:

$$\mathrm{Li}_2(\frac{-1}{n^2-1}) - \frac{-1}{n^2-1} - \frac{1}{4}\frac{1}{(n^2-1)^2}$$

and add back a correction:

$$-\Omega(1) + \frac{1}{4}\Omega(2) = \frac{\pi^2}{48} - \frac{59}{64}$$

where Ω is the zeta-like function

$$\Omega(m) := \sum_{n=2}^{\infty} \frac{1}{(n^2-1)^m}.$$

A careful Eulerian partial fraction decomposition (as detailed in [5]) produces a closed form for all integral $m > 0$

$$\Omega(m) = (-1)^m \{ \sum_{k=1}^{\lfloor m/2 \rfloor} 2^{k+1-2m} \binom{2m-k-1}{m-1} \zeta(2k) - \frac{\binom{2m}{m}}{2^{2m+1}} - \frac{1}{2} \}.$$

Moreover,

$$\sum_{n=1}^{\infty} \binom{n}{1} \{\zeta(2n) - 1\} = \Omega(1) + \Omega(2),$$

and for $m = 2, 3, \cdots$, one may show inductively using another partial fraction argument

$$\sum_{n=1}^{\infty} \binom{n}{m} \{\zeta(2n) - 1\} = \Omega(m) + \Omega(m+1) - \{\zeta(2m) - 1\},$$

from which we may easily obtain a closed form for

$$\sum_{k=1}^{\infty} \mathcal{P}(k)\{\zeta(2k) - 1\} = \sum_{k=1}^{\infty} \mathcal{P}(k)\zeta(2k, 1)$$

for any polynomial $\mathcal{P}$. Thus,

$$\begin{aligned}
\sum_{k=1}^{\infty} \binom{k}{0} \zeta(2k,1) &= \frac{3}{4}, \\
\sum_{k=1}^{\infty} \binom{k}{1} \zeta(2k,1) &= \frac{1}{16} + \frac{1}{2}\zeta(2), \\
\sum_{k=1}^{\infty} \binom{k}{2} \zeta(2k,1) &= \frac{31}{32} - \zeta(4) + \frac{1}{8}\zeta(2), \\
\sum_{k=1}^{\infty} \binom{k}{3} \zeta(2k,1) &= \frac{261}{256} - \zeta(6) + \frac{1}{8}\zeta(4) - \frac{1}{16}\zeta(2), \\
\sum_{k=1}^{\infty} \binom{k}{4} \zeta(2k,1) &= \frac{505}{512} - \zeta(8) - \frac{1}{32}\zeta(4) + \frac{5}{128}\zeta(2),
\end{aligned}$$

and

$$\sum_{k=1}^{\infty} \binom{k}{5} \zeta(2k,1) = \frac{2069}{2048} - \zeta(10) + \frac{1}{32}\zeta(6) + \frac{1}{128}\zeta(4) - \frac{7}{256}\zeta(2)$$

with direct analogues for $\zeta(2k, m)$ for $m > 1$.

To complete this section we observe that for $m = 1, 2, \cdots$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n) - 1}{n^m} = \sum_{n=2}^{\infty} \mathrm{Li}_m(1/n^2).$$

Here, unlike Lemma 2(b) in which $m = 1$, the right-hand sum does not appear to telescope. This identity is easy to verify.

4 EXPLICIT CONTINUED FRACTIONS

It is remarkable that, even though a random fraction's limiting geometric mean exists and furthermore equals the Khinchin constant with probability one, not a single explicit continued fraction has been demonstrated to have geometric mean K_0. Likewise for any negative integer p, not a single explicit fraction has been shown to have Holder mean equal to K_p. In any event, it is worthwhile to mention some explicit fractions with respect to this theoretical impasse.

The continued fraction for e is

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, ...].$$

The elements are eventually comprised of a meshing of two arithmetic progressions, one of which has zero common difference while the other has difference two and diverges. Thus the meshing has diverging geometric mean. Thus e does not possess geometric mean K_0. The harmonic mean for e does exist, but equals $3/2$ which is not K_{-1}. It turns out that any fraction with elements lying in a single arithmetic progression can be evaluated in terms of special functions. Explicitly, for any positive integers a, d

$$[a, a + d, a + 2d, a + 3d, ...] = \frac{I_{a/d-1}(\frac{2}{d})}{I_{a/d}(\frac{2}{d})}$$

where I_ν is the *modified Bessel function* of order ν. These arithmetic progression fractions are certainly interesting, and not beyond deep analysis. It was known, for example, to C. L. Siegel that these fractions are transcendental [17]. But each such fraction has diverging geometric mean and indeed diverging Hölder means. Note that the means are monotone non-decreasing in p and so a fraction with lim inf of its elements infinite has infinite means.

Another example of interest is π, whose continued fraction expansion is

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, ...]$$

The continued fraction elements do not appear to follow any pattern and are widely suspected to be in some sense random. Based on the first 17 million digits, the geometric mean is 2.686393 and the harmonic mean is 1.745882 [10]. These values are reasonably close to K_0 and K_{-1}, but of course no conclusion can be drawn beyond this.

It is a well known theorem of Lagrange that the elements of a simple continued fraction form an eventually periodic sequence if and only if the fraction is an irrational quadratic surd. All Khinchin means K_p for $p = 0, -1, -2, \ldots$ then exist, and are completely determined by the mean of one period of elements. Hence each Khinchin mean of a quadratic surd is an algebraic number. Clearly, for any algebraic number $c = a^{1/b}$ formed from integers a, b, one can write down a quadratic surd having geometric mean c. Along these lines, it is not hard to show that if there exists an integer $m > 2$ such that

$$\frac{\log(K_0/m)}{\log(2/m)}$$

is rational, then there exists a quadratic surd with geometric mean K_0. Thus the issue of transcendence for K_0 and related numbers is an interesting one, and one we return to in the next section. It is also of interest that Liouville transcendentals form a residual null set, and thus comprise a "numerous" set whose members typically do not respect the K_0 limit.

If one were in possession of K_0 to arbitrary accuracy, one could of course construct a fraction having geometric mean K_0 by appending a "2" (respectively, "3") to the element list whenever the current geometric mean were above (below) K_0. There seems to be no way to determine a *priori* the value of, say, the n-th element. Thus such a constructed fraction is not explicit.

However, we propose here, without recourse to the value of K_0, a continued fraction, possessed of explicitly defined elements a_n, that should have geometric mean K_0. Our construction is based on a deterministic stochastic sampling of the Gauss-Kuz'min density, and proceeds as follows. First, for non-negative integer n define the *van der Corput discrepancy sequence* associated with n to be the base-2 number

$$d(n) = 0.b_0 b_1 b_2 \ldots$$

where the b_i are the binary bits of n, with b_0 being least significant. As n runs through positive integers, the sequence of $d(n)$ is confined to (0,1) and has appealing pseudorandom properties. The construction of the number we shall call Z_2 then starts with $a_0 := 0$, and loops as follows:

For $n = 1$ to ∞, set $a_n := \lfloor 1/(2^{d(n)} - 1) \rfloor$

The continued fraction elements a_n thus determined start out:

$$Z_2 = [0, 2, 5, 1, 11, 1, 3, 1, 22, 2, 4, 1, 7, 1, 2, 1, 45, 2, 4, 1, 8, 1, 3, 1, 14, 1, 3, 1, 6,]$$

while the numerical value of Z_2 is approximately

$$Z_2 = 0.46107049595671951935414986933669968767808281325747\ldots.$$

On the basis of anticipated statistical features of our construction we are moved to posit:

Conjecture. The geometric mean of the number Z_2 is in fact the Khinchin constant K_0. Furthermore, every p-th Hölder mean of Z_2 for $p = -1, -2, \ldots$ is the respective Khinchin mean K_p.

There are some features of the Z_2 construction that lend credence to our conjecture. On the basis of the known density properties of the discrepancy set, it can be shown that the elements of Z_2 are unbounded, and that every possible integer element value $a_n = k > 0$ is attained infinitely often.

With regards to the above conjecture, S. Plouffe [14] has reported a computation of the geometric and harmonic means through 5206016 continued fraction elements of Z_2. His results are 2.6854823207 and 1.7454074435, respectively, which are remarkably close to the expected theoretical values. These results raise a deeper question: what is the rate of convergence of these empirical means to their limiting values? The authors have also been informed by T. Wieting that he has an unpublished proof of the basic conjecture, i.e. that the limiting Holder means of Z_2 exist for $p = 0, -1, -2, \ldots$ and furthermore equal the corresponding Khinchin means [20].

In the numerical evaluations reported above and in the next section, the following definitions of empirical Hölder means were used:

$$\begin{aligned} H_0 &:= \left(\prod_{i=1}^{N} a_i\right)^{1/N} \\ H_p &:= \left(\frac{1}{N}\sum_{i=1}^{N} a_i^p\right)^{1/p} \qquad p < 0. \end{aligned}$$

Here H_p is the quantity that should, as $N \to \infty$, converge to K_p with probability one.

5 COMPUTATION OF KHINCHIN MEANS

The authors have explicitly computed K_0 and K_{-1} to more than 7350 decimal digit accuracy. These computations were performed with the aid of the MPFUN multiprecision software [2, 3], which was found to be significantly faster for our purposes than other available multiprecision facilities. One utilizes this software by writing ordinary Fortran-90 code, with multiprecision variables declared to be of type `mp_integer`, `mp_real` or `mp_complex`. In the computations described below, the level of precision was sufficiently high that the "advanced" routines of the Fortran-90 MPFUN library were employed. These routines employ special algorithms, including *fast Fourier transform* (FFT) multiplication, which are efficient for extra-high levels of precision.

K_0 was computed using the formula given above in Theorem 3, with the free integer parameter $N = 100$, and with $N = 120$ as a check. The implementation of this formula was straightforward except for the computation of the Riemann zeta function. To obtain 7350 digit accuracy in the final result, 2048 terms of the indicated series were evaluated, which requires $\{\zeta(2k),\ 0 \le k \le 2048\}$ to be computed. One approach to compute these zeta function values is to apply formulas due to P. Borwein [6]. These formulas are very efficient for computing one or a few zeta function values, but when many values are required as in this case, another approach was found to be more efficient. This method is based on an observation that has previously been used in numerical approaches to Fermat's "Last Theorem" [7, 8]; namely,

$$\begin{aligned}
\coth(\pi x) &= \frac{-2}{\pi x}\sum_{k=0}^{\infty}\zeta(2k)(-1)^k x^{2k} \\
&= \cosh(\pi x)/\sinh(\pi x) \\
&= \frac{1}{\pi x}\cdot\frac{1+(\pi x)^2/2!+(\pi x)^4/4!+(\pi x)^6/6!+\cdots}{1+(\pi x)^2/3!+(\pi x)^4/5!+(\pi x)^6/7!+\cdots}.
\end{aligned}$$

Let $N(x)$ and $D(x)$ be the numerator and denominator polynomials obtained by truncating these two series to n terms. Then the approximate reciprocal $Q(x)$ of $D(x)$ can be obtained by applying the Newton iteration

$$Q_{k+1}(x) := Q_k(x) + [1 - D(x)Q_k(x)]Q_k(x).$$

Once $Q(x)$ has been computed to sufficient accuracy, the quotient polynomial is simply the product $N(x)Q(x)$. The required values $\zeta(2k)$ can then be obtained from the coefficients of this polynomial.

Computation time for the Newton iteration procedure can be reduced by starting with a modest polynomial length and precision level, iterating to convergence, doubling each, etc., until the final length and precision targets are achieved. Computation time can be further economized by performing the two polynomial multiplications indicated in the above formula using a FFT-based convolution scheme. In our implementation, FFTs were actually performed at two levels of this computation: (i) to multiply pairs of polynomials, where the data elements to be transformed are the multiprecision polynomial coefficients, and (ii) to multiply pairs of multiprecision numbers, where the data elements to be transformed are integers representing successive sections of the binary representations of the two multiprecision numbers.

K_{-1} was computed by applying the formula in Corollary 7. Again, the challenge here is to pre-compute values of the Riemann zeta function for integer values. But in this case both odd and even values are required. The odd values can be economically computed by applying the following two formulas, the first given by Ramanujan, but simplified slightly, the second derived by differentiating a companion identity of Ramanujan ([4]):

$$\begin{aligned}
\zeta(4N+3) &= -2\sum_{k=1}^{\infty}\frac{1}{k^{4N+3}(\exp(2k\pi)-1)} \\
&\quad -\pi(2\pi)^{4N+2}\sum_{k=0}^{2N+2}(-1)^k\frac{B_{2k}B_{4N+4-2k}}{(2k)!(4N+4-2k)!} \\
\zeta(4N+1) &= -\frac{1}{N}\sum_{k=1}^{\infty}\frac{(2\pi k+2N)\exp(2\pi k)-2N}{k^{4N+1}(\exp(2k\pi)-1)^2} \\
&\quad -\frac{1}{2N}\pi(2\pi)^{4N}\sum_{k=1}^{2N+1}(-1)^k\frac{B_{2k}B_{4N+2-2k}}{(2k-1)!(4N+2-2k)!}.
\end{aligned}$$

Here B_{2k} is as always the $2k$-th Bernoulli number.

Alternatively the formulas can be written in terms of the even zetas as

$$\begin{aligned}
\zeta(4N+3) &= -2\sum_{k=1}^{\infty}\frac{1}{k^{4N+3}(\exp(2k\pi)-1)} \\
&\quad +\frac{1}{\pi}\{\frac{(4N+7)}{2}\zeta(4N+4)-\sum_{k=1}^{N}2\zeta(4k)\zeta(4N+4-4k)\} \\
\zeta(4N+1) &= -\frac{1}{N}\sum_{k=1}^{\infty}\frac{(2\pi k+2N)\exp(2\pi k)-2N}{k^{4N+1}(\exp(2k\pi)-1)^2} \\
&\quad +\frac{1}{2N\pi}\{\sum_{k=1}^{2N}(-1)^k 2k\zeta(2k)\zeta(4N+2-2k)+(2N+1)\zeta(4N+2)\}.
\end{aligned}$$

These two formulas are not very economical for computing a single odd value or just a few odd values of $\zeta(k)$ — again, the formulas in [6] are more efficient for such purposes. But these Ramanujan formulas are quite efficient when a large number of odd zetas are required. Note that the infinite series in the two formulas can be inexpensively evaluated for many N simultaneously, since the expensive parts of these expressions do not involve N. Further, the evaluation of the infinite series can be cut off once terms for a given N are smaller than the "epsilon" of the numeric precision level being used. Happily, convergence here is fairly rapid for large N.

At first glance, the latter summations in these two formulas may appear quite expensive to evaluate. But note that each is merely the polynomial product of two vectors consisting principally of even zeta values. Thus both sets of summation results can be computed using multiprecision FFT-based convolutions.

Computation of K_0 to 7350 digit precision required 2.5 hours on an IBM RS6000/590 workstation, and computation of K_{-1} required some 12 hours. Excerpts of the resulting decimal expansions for each are included in the appendix. The complete expansions are available from the authors.

One intriguing question is whether the continued fraction expansions of the Khinchin constants themselves satisfy the geometric and harmonic mean conditions of their definitions. We found that the geometric and harmonic means for our value of K_0 were 2.663660 and 1.746398, respectively. The geometric and harmonic means for our value of K_{-1} were 2.723115 and 1.746965, respectively. The issue of where to terminate the list of continued fraction elements is an interesting one. We employed a simple criterion: if x is known numerically, to D decimals to the right of the decimal point, generate continued fraction elements for x until a convergent p/q has $2q^2 > 10^D$. The reason for choosing this simple criterion is the theorem that at least one of any two successive convergents must satisfy

$$|\frac{p}{q} - x| < \frac{1}{2q^2}$$

and conversely, any reduced ratio p/q satisfying this inequality must be a convergent of x ([11]).

To give statistical perspective to these results, we computed these same empirical means for 100 pseudorandom multiprecision numbers of the same precision. The average and standard deviation of their geometric means were 2.683740 and 0.030124, respectively. The same statistics for their harmonic means were 1.745309 and 0.011148, respectively. Note that these two averages are in good agreement with the theoretical values K_0 and K_{-1}. In any event, it appears that the empirical geometric and harmonic means for K_0 and K_{-1} are within reasonable statistical limits of the expected theoretical values.

A question implicitly asked in the previous section is whether K_0 or K_{-1} is algebraic. This question can be numerically explored by means of integer relation algorithms. A vector of real numbers $(x_1, x_2, \cdots, x_n)$ is said to possess an *integer relation* if there exist integers a_k such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$. It can easily be seen that a real number α is algebraic of degree $n-1$ if and only if the vector $(1, \alpha, \alpha^2, \cdots, \alpha^{n-1})$ possesses an integer relation. Even if α is not algebraic, integer relation algorithms produce bounds that allow one to exclude relations within a region.

We employed the "PSLQ" algorithm developed by Ferguson and one of the authors, a simplified version of which is given in [1]. This algorithm, when applied to power vectors generated from our computed values of K_0 and K_{-1}, found no relations for either. On the contrary, we obtained the following result: neither K_0 or K_{-1} satisfies a polynomial of the form

$$0 = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \cdots + a_{50}\alpha^{50}$$

with integer coefficients a_k of absolute value 10^{70} or less.

In a second experiment, we explored the possibility that K_0 or K_{-1} is given by a multiplicative formula involving powers of primes and some well-known mathematical constants. To that end, let p_k denote the k-th prime. We established, using PSLQ, that neither K_0 or K_{-1} satisfies a relation of the form

$$\begin{aligned} 0 &= a_0 \log\alpha + \sum_{k=1}^{15} a_k \log p_k \\ &\quad + a_{16}\log\pi + a_{17}\log e + a_{18}\log\gamma + a_{19}\log\zeta(3) + a_{20}\log\log 2 \end{aligned}$$

with integer coefficients a_k of absolute value 10^{20} or less. By exponentiating this expression, it follows that neither K_0 or K_{-1} satisfies a corresponding multiplicative formula with exponents of absolute value 10^{20} or less.

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This software and documentation, as well as that described in [2], may be obtained by sending electronic mail to `mp-request@nas.nasa.gov`, or by using Mosaic at address `http://www.nas.nasa.gov`.

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Appendix: The Khinchin Constant K_0 to 7,350 Digits

2.
68545200106530644530971483548179569382038229399446
29530511523455572188595371520028011411749318476979
95153465905288090082897677716410963051792533483259
66838185231542133211949962603932852204481940961806
86641664289308477880620360737053501033672633577289
04990427070272345170262523702354581068631850103237
46558037750264425248528694682341899491573066189872
07994137235500057935736698933950879021244642075289
74145914769301844905060179349938522547040420337798
56398310157090222339100002207725096513324604444391
.....
36909874406573435125594396103980583983755664559601

The Khinchin Harmonic Mean K_{-1} to 7,350 Digits

1.
74540566240734686349459630968366106729493661877798
42565950137735160785752208734256520578864567832424
20977343982577985596531102601834294460206578713176
15026238960612981165718728271638949622593992929776
06160830078357479801549029312671643067241248453710
96077711207484391474195803753220015690822609477078
44894635568203493582068440202422591615018316479048
29229656977733143662210991806388842581650599997697
61391683577259217628635718712601565066754443340174
00283376465305136584406098398017126202832041200630
.....
78553128249666473680304034761497467330708479436280

Khinchin Means K_p for Various Negative p to 50 Digits

p	K_p
-2	1.450340328495630406052983076680697881408299979605904...
-3	1.313507078687985766717339447072786828158129861484792...
-4	1.236961809423730052626227244453422567420241131548937...
-5	1.189003926465513154062363732771403397386092512639671...
-6	1.156552374421514423152605998743410046840213070718761...
-7	1.133323363950865794910289694908868363599098282411797...
-8	1.115964408978716690619156419345349695769491182230400...
-9	1.102543136670728013836093402522568351022221284149318...
-10	1.091877041209612678276110979477638256493272651429656...

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